Universal Helical Geometry: Spherical Coordinates, Earth Rotation, and DNA Topology

Oliviana Ursuleac

16 January 2026

Abstract

This paper derives the complete transformation of a standard helix $\mathbf{r}(t) = (R\cos t, R\sin t, ct)$ from Cartesian to spherical coordinates (r, θ, φ), providing explicit formulas for $r(t)$, $\theta(t)$, $\varphi(t)$, and their derivatives. We establish the constant angular velocity $\theta'(t) = 1$ through rigorous calculation using the two-argument arctangent and Cartesian velocity components. The analysis reveals the helix's uniform rotation about the z-axis with monotonic polar angle evolution, connecting to Cuculescu's [1] geometric constructions of helical motion on developable surfaces.

1 Introduction

A circular helix is parametrized in Cartesian coordinates by:

$$\mathbf{r}(t) = (R\cos t, R\sin t, ct), \quad t \in \mathbb{R} \tag{1}$$

where $R > 0$ is the radius and $c = p/(2\pi)$ determines the pitch p (axial advance per full turn). This curve lies on the cylinder $x^2 + y^2 = R^2$ and exhibits helical symmetry under screw displacements.

Spherical coordinates (r, θ, φ) relate to Cartesian coordinates via:

$$x = r\sin\varphi\cos\theta \tag{2}$$

$$y = r\sin\varphi\sin\theta \tag{3}$$

$$z = r\cos\varphi \tag{4}$$

with $r \geq 0$, $\theta \in [0, 2\pi)$, $\varphi \in [0, \pi]$. Our goal is the explicit transformation $\mathbf{r}(t) \mapsto (r(t), \theta(t), \varphi(t))$.

2 Calculation of Radial Distance $r(t)$

The radial distance is:

$$r(t) = \sqrt{x^2(t) + y^2(t) + z^2(t)} \tag{5}$$

Substitute helix coordinates:

$$\begin{aligned} r(t)^2 &= R^2\cos^2 t + R^2\sin^2 t + (ct)^2 \\ &= R^2(\cos^2 t + \sin^2 t) + c^2t^2 = R^2 + c^2t^2 \end{aligned}$$

Thus:

$$\boxed{r(t) = \sqrt{R^2 + c^2t^2}} \tag{6}$$

Note that $r(t) \to \infty$ as $|t| \to \infty$, reflecting the unbounded nature of the helix.

3 Derivation of Azimuth $\theta(t)$

The azimuth is defined by the two-argument arctangent:

$$\theta(t) = \operatorname{atan2}(y(t), x(t)) = \operatorname{atan2}(R\sin t, R\cos t) \tag{7}$$

3.1 Trigonometric Reduction

Consider the tangent:

$$\tan\theta(t) = \frac{y(t)}{x(t)} = \frac{R\sin t}{R\cos t} = \tan t \tag{8}$$

Since $R > 0$ is constant and $\sqrt{x^2 + y^2} = R$, the atan2 function preserves the quadrant. For $(x, y) = (R\cos t, R\sin t)$, we have $\operatorname{atan2}(R\sin t, R\cos t) = t$ on the principal branch.

3.2 Principal Continuous Branch

Choosing the branch with $\theta(0) = 0$ and requiring continuity:

$$\boxed{\theta(t) = t, \quad t \in \mathbb{R}} \tag{9}$$

More generally, $\theta(t) = t + 2k\pi$ for $k \in \mathbb{Z}$, but we use the principal value.

4 Polar Angle $\varphi(t)$

The polar angle satisfies:

$$\cos\varphi(t) = \frac{z(t)}{r(t)} = \frac{ct}{\sqrt{R^2 + c^2t^2}} \tag{10}$$

For $c > 0$ (right-handed helix):

- When $t > 0$: $\cos\varphi > 0$, so $\varphi \in (0, \pi/2)$
- When $t = 0$: $\cos\varphi = 0$, so $\varphi = \pi/2$
- When $t < 0$: $\cos\varphi < 0$, so $\varphi \in (\pi/2, \pi)$

Thus:

$$\boxed{\varphi(t) = \arccos\left(\frac{ct}{\sqrt{R^2 + c^2t^2}}\right)} \tag{11}$$

As $t \to +\infty$, $\varphi(t) \to \arccos(1) = 0$ (approaching north pole). As $t \to -\infty$, $\varphi(t) \to \arccos(-1) = \pi$ (approaching south pole).

5 Angular Velocity $\theta'(t)$

5.1 Direct Differentiation

Since $\theta(t) = t$:

$$\theta'(t) = 1 \tag{12}$$

This constant angular velocity reflects the uniform rotation about the z-axis.

5.2 Cartesian Verification

The general formula for azimuthal velocity is:

$$\theta'(t) = \frac{x(t)y'(t) - y(t)x'(t)}{x^2(t) + y^2(t)} \tag{13}$$

Compute derivatives:

$$x'(t) = -R\sin t, \qquad y'(t) = R\cos t$$

Numerator:

$$\begin{aligned} xy' - yx' &= (R\cos t)(R\cos t) - (R\sin t)(-R\sin t) \\ &= R^2\cos^2 t + R^2\sin^2 t = R^2 \end{aligned}$$

Denominator:

$$x^2 + y^2 = R^2 \tag{14}$$

Thus:

$$\boxed{\theta'(t) = \frac{R^2}{R^2} = 1} \tag{15}$$

Both methods confirm the constant azimuthal velocity.

6 Polar Angle Derivative $\varphi'(t)$

Let $u(t) = \cos\varphi(t) = \frac{ct}{\sqrt{R^2+c^2t^2}}$.

Using the chain rule $\frac{d}{dt}\arccos(u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dt}$:

Step 1: Compute $\frac{du}{dt}$

$$\begin{aligned} \frac{du}{dt} &= \frac{d}{dt}\left[ct(R^2+c^2t^2)^{-1/2}\right] \\ &= c(R^2+c^2t^2)^{-1/2} + ct\cdot\left(-\frac{1}{2}\right)(R^2+c^2t^2)^{-3/2}\cdot 2c^2t \\ &= c(R^2+c^2t^2)^{-1/2} - c^3t^2(R^2+c^2t^2)^{-3/2} \\ &= \frac{c(R^2+c^2t^2) - c^3t^2}{(R^2+c^2t^2)^{3/2}} = \frac{cR^2}{(R^2+c^2t^2)^{3/2}} \end{aligned}$$

Step 2: Compute $\sqrt{1-u^2}$

$$1 - u^2 = 1 - \frac{c^2t^2}{R^2+c^2t^2} = \frac{R^2}{R^2+c^2t^2} \quad\Rightarrow\quad \sqrt{1-u^2} = \frac{R}{\sqrt{R^2+c^2t^2}}$$

Step 3: Apply chain rule

$$\begin{aligned} \varphi'(t) &= -\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dt} \\ &= -\frac{\sqrt{R^2+c^2t^2}}{R}\cdot\frac{cR^2}{(R^2+c^2t^2)^{3/2}} \\ &= -\frac{cR^2}{R(R^2+c^2t^2)} = -\frac{cR}{R^2+c^2t^2} \end{aligned}$$

For $c > 0$ and $R > 0$:

$$\boxed{\varphi'(t) = -\frac{cR}{R^2+c^2t^2} < 0} \tag{16}$$

The negative sign indicates monotonic decrease: the helix spirals from south to north (for $c > 0$) as t increases. Note that $\varphi'(t) \to 0$ as $|t| \to \infty$, meaning the rate of polar change slows as the helix extends.

Quantity	Formula	Property
$r(t)$	$\sqrt{R^2 + c^2t^2}$	Unbounded as $\|t\| \to \infty$
$\theta(t)$	t	Uniform rotation
$\varphi(t)$	$\arccos\left(\frac{ct}{\sqrt{R^2+c^2t^2}}\right)$	Monotonic decrease
$\theta'(t)$	1	Constant
$\varphi'(t)$	$-\frac{cR}{R^2+c^2t^2}$	Negative, $\to 0$ as $\|t\| \to \infty$

Table 1: Complete helical coordinates summary

7 Application: Earth's Rotation as Helical Motion

Earth's rotation provides a natural realization of helical geometry. Consider Earth as an oblate spheroid rotating about its polar axis with angular velocity $\mathbf{\Omega} = (0, 0, \omega_\oplus)$, where $\omega_\oplus = 7.292115 \times 10^{-5}$ rad/s (sidereal day).

7.1 Parametrization of Surface Motion

A point P on Earth's surface at latitude $\lambda \in [-90°, 90°]$ traces:

$$\mathbf{r}_P(t) = (R_\oplus \cos\lambda \cos(\omega_\oplus t), R_\oplus \cos\lambda \sin(\omega_\oplus t), R_\oplus \sin\lambda) \tag{17}$$

where $R_\oplus \approx 6371$ km is mean radius. In spherical coordinates:

$$r(t) = R_\oplus = \text{constant}, \tag{18}$$

$$\theta(t) = \omega_\oplus t \pmod{2\pi}, \tag{19}$$

$$\varphi(t) = \frac{\pi}{2} - \lambda = \text{constant}. \tag{20}$$

Thus $\theta'(t) = \omega_\oplus$ (uniform azimuthal rotation), $\varphi'(t) = 0$.

7.2 Helical Flows and the Geodynamo

Helical motion plays a fundamental role in generating Earth's magnetic field. In the outer core's conducting fluid, rotating convection induces nonzero kinetic helicity:

$$\mathcal{H} = \langle \mathbf{v} \cdot (\nabla \times \mathbf{v}) \rangle \neq 0, \tag{21}$$

which is a necessary condition for the α-effect in mean-field magnetohydrodynamics [2, 3, 4]. Under the First-Order Smoothing Approximation (FOSA)—which assumes homogeneous, isotropic turbulence with correlation time much shorter than large-scale field evolution—the turbulent electromotive force yields:

$$\alpha \sim -\frac{\tau}{3} \langle \mathbf{v} \cdot (\nabla \times \mathbf{v}) \rangle, \tag{22}$$

where τ is the eddy correlation time.

Here, $v_\perp$ denotes the magnitude of the horizontal (azimuthal) velocity component of convective plumes. This ensures that the helical motion is well-defined throughout the outer core and not just along a single trajectory.

These screw-like flows arise from Earth's rotation plus radial convection, matching our helix velocity field $\mathbf{v}(t) = v_z\hat{\mathbf{z}} + v_\perp(\cos(\omega t), \sin(\omega t), 0)$. The existence of nonzero helicity is **strictly necessary** for large-scale dynamo action.

7.3 Parameter Comparison

This connects planetary dynamics to microscopic helical structures through universal screw symmetry $\mathbb{R} \ltimes S^1$, with Earth's geodynamo as a macroscopic realization.

Parameter	Earth Value	Helix Analog
Radius R	6371 km	Cylinder radius
Angular speed ω	7.29×10^{-5} rad/s	$\theta'(t)$
Pitch parameter c	0.465 km/s	Axial convection
Coriolis f	$2\omega_{\oplus} \sin\lambda$	Handedness control
Helicity $\mathcal{H}$	> 0 (right-handed)	$\mathbf{v} \cdot \boldsymbol{\omega} > 0$

Table 2: Earth rotation vs. mathematical helix

8 Application: DNA Double Helix Geometry

The B-DNA double helix provides a biological realization of helical geometry, with sugar-phosphate backbones tracing counter-rotating right-handed helices around a common axis.

8.1 Geometric Parametrization of DNA

B-DNA structure [5, 6] has typical parameters:

- Helix radius $R \approx 10$ Å (to C1' atoms)
- Helix pitch $p \approx 34$ Å (10 base pairs per turn)
- Twist per base pair $\omega = 36^\circ \approx 2\pi/10.4$ rad/bp
- Rise per base pair $c = 3.4$ Å/bp

A parametric equation for one backbone strand (strand 1) is:

$$\mathbf{r}_{\mathrm{DNA},1}(t) = \big(R\cos(\omega t),\ R\sin(\omega t),\ ct\big), \quad t \in \mathrm{bp}. \tag{23}$$

The complementary strand (strand 2), being antiparallel, traces:

$$\mathbf{r}_{\mathrm{DNA},2}(t) = \big(R\cos(-\omega t + \pi),\ R\sin(-\omega t + \pi),\ ct\big) = \big(-R\cos(\omega t),\ R\sin(\omega t),\ ct\big), \tag{24}$$

which corresponds to azimuth $\theta_2(t) = -\omega t + \pi$ (counter-rotation with phase shift). Note that strand 1 is right-handed while strand 2 is antiparallel, effectively forming a counter-rotating helix. This preserves the double-helical topology and ensures the linking number and twist calculations remain consistent.

8.2 Spherical Coordinate Representation

Using the transformation derived for a general helix (strand 1):

$$r_{\mathrm{DNA}}(t) = \sqrt{R^2 + (ct)^2} \tag{25}$$

$$\theta_{\mathrm{DNA}}(t) = \omega t \pmod{2\pi} \tag{26}$$

$$\varphi_{\mathrm{DNA}}(t) = \arccos\left(\frac{ct}{\sqrt{R^2 + (ct)^2}}\right) \tag{27}$$

The angular velocity is constant:

$$\theta'_{\mathrm{DNA}}(t) = \omega \approx 0.604 \text{ rad/bp}. \tag{28}$$

8.3 Topological Invariants

The DNA double helix has well-defined topological invariants:

$$Lk = Tw + Wr, \tag{29}$$

where Lk is the linking number, Tw is the total twist, and Wr is the writhe. For relaxed B-DNA, $Lk_0 = n/h$ where n is the number of base pairs and $h \approx 10.4$ bp/turn. Underwound DNA (e.g., from negative supercoiling) has $Lk < Lk_0$; a typical example shows:

$$\Delta Lk \approx -1.74 \text{ turns per 100 bp.} \tag{30}$$

Helical symmetry is preserved under screw displacements that match the pitch-rise ratio.

8.4 Physical Helicity in DNA

Charge transport along the phosphate backbone and base-pair stacking exhibit helical features. Locally, one can define a current helicity:

$$\mathcal{H}_{\text{DNA}} = \mathbf{j} \cdot (\nabla \times \mathbf{j}) > 0, \tag{31}$$

analogous to the kinetic helicity in planetary dynamos. Base stacking maintains orientation perpendicular to the helix axis, reinforcing the screw symmetry.

8.5 Unified Helical Symmetry Across Scales

System	**Scale**	**Radius R**	**Freq. ω**	**Physical Role**
DNA	10 Å	10 Å	0.604 rad/bp	Genetic information storage
Atomic	1–5 fm	R_{Fermi}	$\hbar/mc$	Spin-orbit coupling
Earth	6371 km	Surface	7.29×10^{-5} rad/s	Geodynamo and rotational dynamics

Table 3: Helical symmetry as a unifying principle across scales.

Thus, DNA realizes the same $\mathbb{R} \ltimes S^1$ screw symmetry as the mathematical helix and Earth's rotation, connecting molecular biology to planetary physics through universal helical geometry.

9 Conclusion

The transformations derived in this paper confirm that a standard helix exhibits uniform azimuthal rotation with monotonic polar evolution, fully captured in spherical coordinates by the relations:

$$r(t) = \sqrt{R^2 + c^2t^2}, \quad \theta(t) = t, \quad \varphi(t) = \arccos \frac{ct}{\sqrt{R^2 + c^2t^2}}.$$

The constant azimuthal velocity $\theta'(t) = 1$ reflects the defining screw symmetry $\mathbb{R} \ltimes S^1$, while the polar evolution $\varphi'(t) = -cR/(R^2 + c^2t^2) < 0$ ensures a monotonic helical ascent along the axis (for $c > 0$).

By applying this framework to Earth, we demonstrated that surface points and core convection flows trace trajectories analogous to helical motion. The existence of nonzero kinetic helicity in these flows is a **necessary condition for the geodynamo**, illustrating how abstract helix geometry directly informs planetary-scale magnetic phenomena.

At the molecular scale, the DNA double helix exhibits the same geometric and topological features, including screw symmetry and helicity. The antiparallel nature of the two strands is captured by counter-rotating helices with a phase shift. Topological invariants such as the linking number, twist, and writhe preserve the structural integrity of DNA under various conformational changes, showing that helical geometry governs both structure and function at the nanoscale.

Taken together, these examples demonstrate that **helical symmetry is a unifying principle across scales**, from subatomic and molecular systems to planetary dynamics. The concept of helicity, whether kinetic or topological, provides a consistent measure of "screw-likeness," linking geometry, physics, and function in natural systems.

This cross-disciplinary perspective suggests that helical motion and its associated invariants are **fundamental organizing principles** in nature, shaping the behavior of systems ranging from DNA and atomic orbitals to planetary cores and stellar structures. Future work could explore additional manifestations of this symmetry in fluid dynamics, astrophysics, and biological macromolecules, potentially revealing deeper connections between geometry, dynamics, and topology in complex systems.

References

[1] Cuculescu, I., Kleitsch, Ș., & Ottescu, C. (1997). *Geometrie în spațiu.* Editura Universității din București. https://biblioteca-digitala.ro/reviste/carte/universitatea-bucuresti/cuculescu_kleitsch_ottescu-geometrie-spatiu_1997.pdf

[2] Moffatt, H. K. (1978). *Magnetic Field Generation in Electrically Conducting Fluids.* Cambridge University Press.

[3] Krause, F., & Rädler, K.-H. (1980). *Mean-Field Magnetohydrodynamics and Dynamo Theory.* Pergamon Press.

[4] Soward, A. M. (1974). A Screw Dynamo in Rotating Fluids. *Journal of Fluid Mechanics*, 64(1), 1–20.

[5] Franklin, R. E., & Gosling, R. G. (1953). Molecular Configuration in Sodium Thymonucleate. *Nature*, 171(4356), 740–741.

[6] Watson, J. D., & Crick, F. H. C. (1953). Molecular Structure of Nucleic Acids. *Nature*, 171(4356), 737–738.

Acknowledgment

The author acknowledges the support of various AIs in the geometric derivation and Latex optimization of this manuscript.

Appendix

A Derivation of Kinetic Helicity and the α-Effect

To connect the geometric helix to planetary dynamics, we define the physical measure of "screw-likeness" in a fluid flow.

A.1 Helicity Density

For a simplified helical flow field with tangent velocity $\mathbf{v} = (-R\sin t, R\cos t, c)$ and vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, one finds that helical flows have **nonzero and constant** helicity density, confirming screw-like motion:

$$h \propto c\,\omega > 0 \quad \text{(for right-handed helices).} \tag{32}$$

A.2 The Dynamo α-Effect

In Earth's core, Coriolis forces generate helical convective plumes. Under the First-Order Smoothing Approximation (FOSA)—valid when turbulent eddies have correlation time τ much shorter than the large-scale magnetic field evolution time, and when the turbulence is statistically homogeneous and isotropic—the turbulent electromotive force $\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle$ yields:

$$\boldsymbol{\mathcal{E}} = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} + \dots, \tag{33}$$

where the α-coefficient is:

$$\alpha = -\frac{\tau}{3} \langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle, \tag{34}$$

and $\mathbf{u}$ is the fluctuating velocity, $\mathbf{b}$ the small-scale magnetic field, $\mathbf{B}$ the mean field, and β is turbulent diffusivity.

A.3 Topological Interpretation

The total helicity

$$\mathcal{H} = \int_V h \, dV \tag{35}$$

is an invariant of ideal (non-dissipative) flows, measuring the **linkage and chirality of vortex lines**. This universal chirality preference connects DNA supercoiling, nuclear spin-orbit coupling, and geodynamo handedness through the fundamental property of helical symmetry.